Puzzling through problems

a habit of mind

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The slides will be available…

On

thinkmath.edc.org

~~735~~ ~~391~~ ~~424~~

163 ~~230~~

hundreds digit > 6

tens digit is
3, 4, or 5

ones digit < 5

the number is
a multiple of 5

the number
is even

the tens digit is
greater than the
hundreds digit

tens digit < ones digit

the ones digit is
twice the tens digit

the number is
divisible by 3

Puzzling things through

- Part of child's world
- Permission to think
- "Pure mathematical thinking" minus content
- But could car

8 year old detectives!

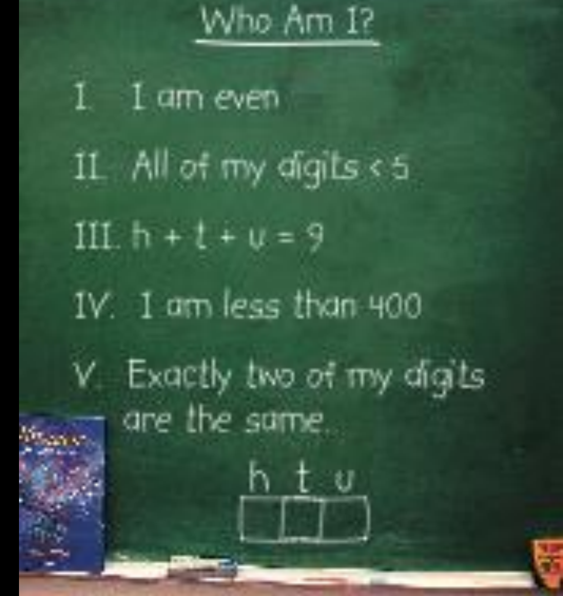
I. I am even.

II. All of my digits < 5

III. $h + t + u = 9$

IV. I am less than 400.

V. Exactly two of my digits are the same.



h	*t*	*u*
1	4	4

h	*t*	*u*
	0	0
1	1	~~1~~
2	2	2
3	3	~~3~~
4	4	4
~~5~~	~~5~~	~~5~~
~~6~~	~~6~~	~~6~~
~~7~~	~~7~~	~~7~~
~~8~~	~~8~~	~~8~~
~~9~~	~~9~~	~~9~~

~~432~~
342
234
324
144
~~414~~

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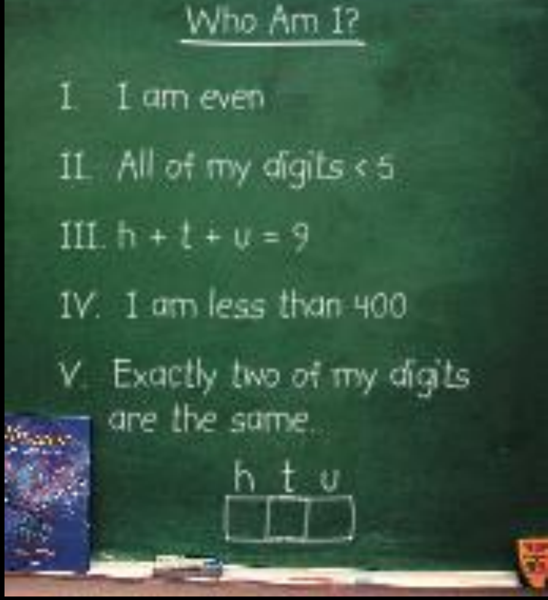

h	*t*	*u*
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	0	0
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2	2	2
3	3	~~3~~
~~4~~	4	4
~~5~~	~~5~~	~~5~~
~~6~~	~~6~~	~~6~~
~~7~~	~~7~~	~~7~~
~~8~~	~~8~~	~~8~~
~~9~~	~~9~~	~~9~~

~~1,1,7~~

~~2,2,5~~

~~3,3,3~~

4,4,1

Four simpler puzzles

Who Am I?

- I am even.
- The sum of my digits is 16.

t	u

0
2
4
6
8

Who Am I?

- I am odd.
- The sum of my digits is 2.

t	u

Who Am I?

- I am even.
- I am a perfect square.
- My units digit is twice my tens digit.

t	u

~~0~~
~~1~~
~~4~~
~~9~~
~~16~~

Who Am I?

- I am odd.
- $u > t$
- The sum of my digits is 10.
- I am less than 30.

t	u

Invent your own! Who Am I?

- I’m a three digit number. Who am I?
- Clues:
  - Blah
  - Blah
  - Blah
  - …

h	t	u

When kids invent their own puzzles,
they invent clues *at their level.*

Good puzzles are ones that are
easy enough to solve,
hard enough to be fun, and
the hard part is the puzzling through,
not the arithmetic.

Why a puzzle?

- Fun (intellectual surprise!)
- Feels smart (intellectual reward)
- *Because it's puzzling*
  - "Problems" are *problems*
  - Puzzles give us permission to think
- Puzzles can easily be adjusted to fit students
- And, because we're not cats

“Tail-less” word problems

Autopilot strategies

We make fun of thought-free "strategies."

Many numbers: $+$

Two numbers close together: $-$ or $\times$

Two numbers, one large, one small: $\div$

The *idea* of a word problem…

- An attempt at *reality*

  A *situation* rather than a "naked" calculation

- The goal is the ***problem***, not the words

  - Necessarily bizarre dialect: low redundancy or *very* wordy

The *idea* of a word problem…

- An attempt at *reality*

  A *situation* rather than a "naked" calculation

"Clothing the naked" with words makes problems that are *linguistically harder without improving the mathematics*.

In *tests* it is discriminatory!

- Tests of language proficiency test *language proficiency*
- Tests of mathematics… *and* language!

Attempts to be efficient (spare)

- Stereotyped wording key words
- Stereotyped structure autopilot strategies

Writers create bizarre wordings with irrelevant numbers, just to confuse kids.

Key words

We rail against key word strategies.

Ben and his sister were eating pretzels.
Ben left 7 of his pretzels.
His sister left 4 of hers.
How many pretzels were left?

So writers do cartwheels to subvert them.

But, frankly, it is *smart* to look for clues!
This is how language *works*!

If the goal is *mathematics* and to teach children to think and communicate clearly…

…deliberately perverting *our* wording to make it *un*clear is not a good model!

So what *can* we do to help students learn to read and interpret story-based problems correctly?

“Headline Stories” thinkmath.edc.org

Less is more!

> Ben and his sister were eating pretzels.
>
> Ben left 7 of his pretzels.
>
> His sister left 4 of hers.

What questions *can* we ask?

Children learn the *anatomy* of problems by creating them. (Neonatal problem posing!)

"Headline Stories"

- Do it yourself!
  Use any word problem you like.

3 Todd sold ornaments at a craft fair. The first customer bought 5 ornaments. The second customer bought half of what Todd had left. The third customer bought 8 ornaments. After that Todd had 2 ornaments left.

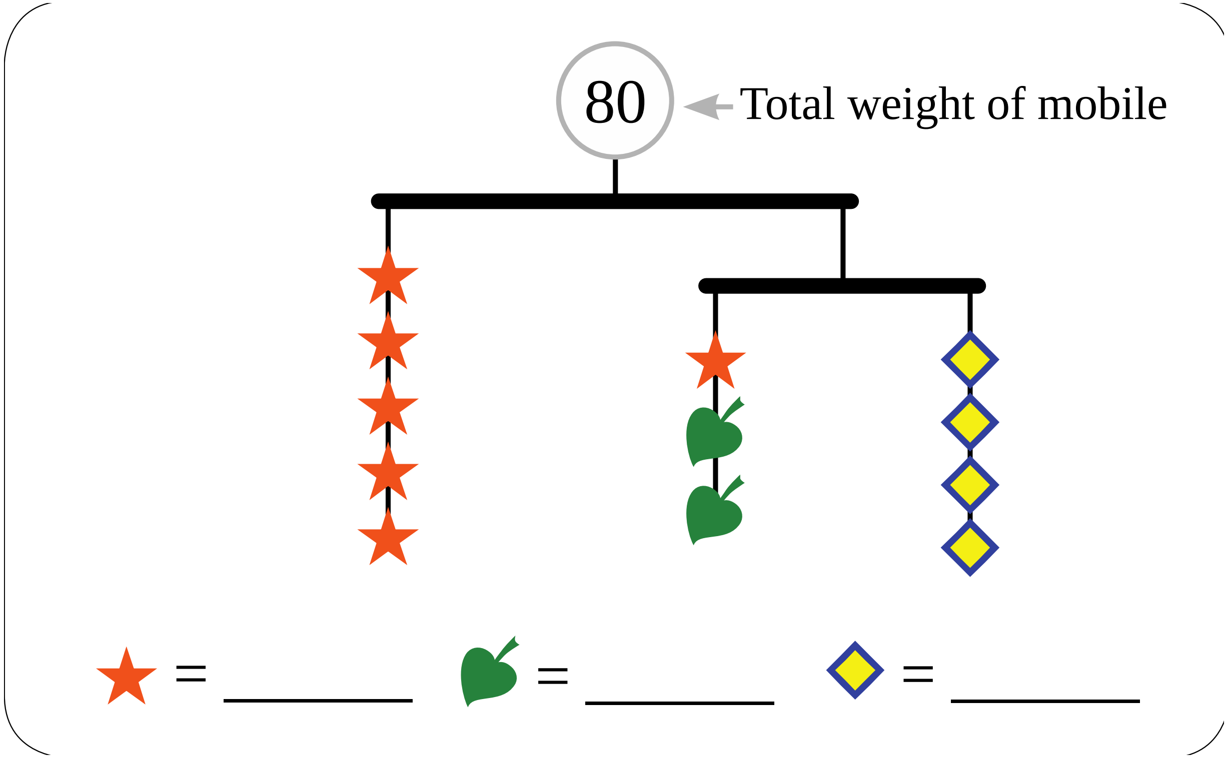
- What *can* I do? What *can* I figure out?
  What *can* a "good" question for this be?

Algebraic puzzles

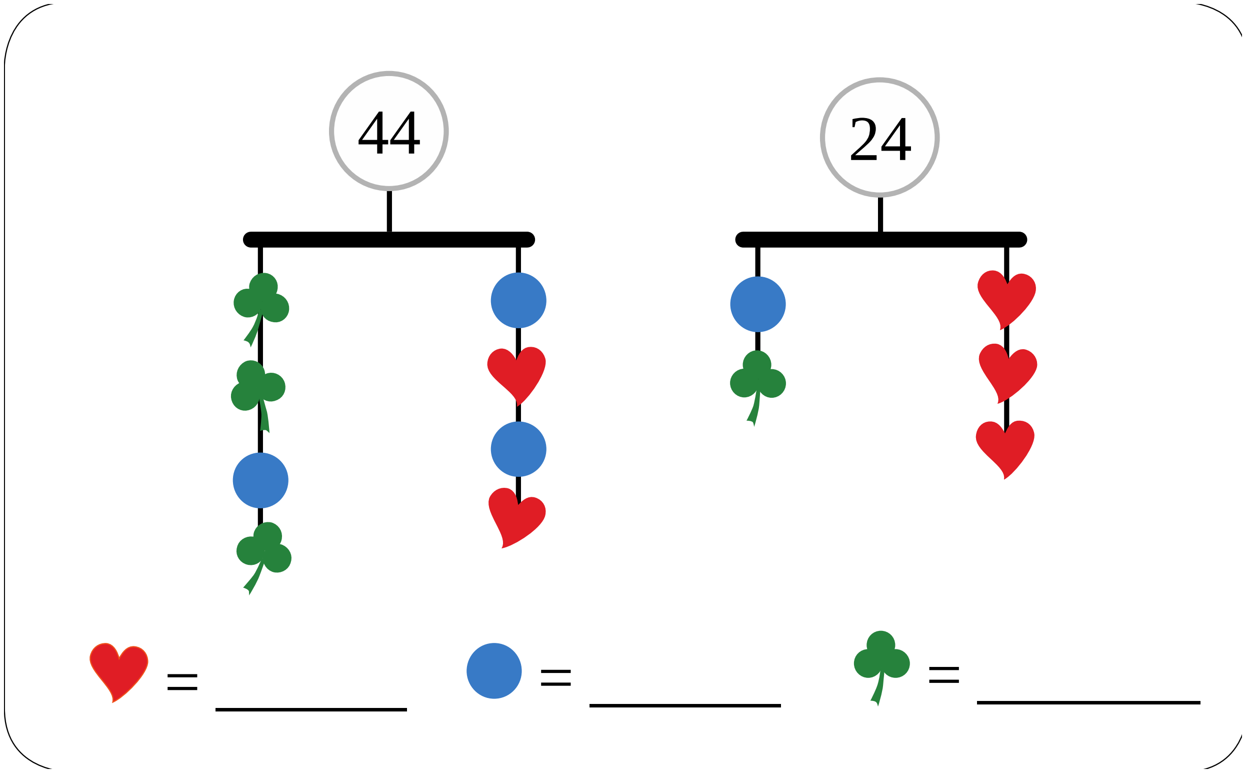
- Who Am I? puzzles (multiple constraints) ✔
- Mobile puzzles (equations and systems)
- KenKen-like puzzles (multiple constraints)
- Think Of a Number tricks (algebraic language)

- And then, to end, a video of “raw practice”

Mobile Puzzles

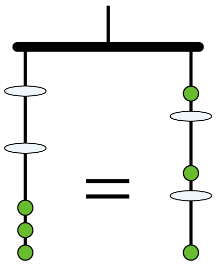


Mobile Puzzles

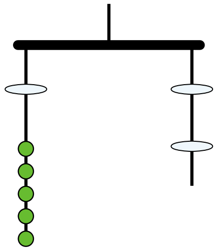


Making the logic of algebra explicit

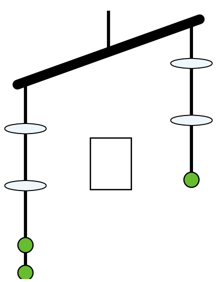
1 Τηισ μοβιλε *always balances.* Ωηψ?



2 Τηισ μοβιλε *only balances when* τηε βυχκετσ ρεπρεσεντ α χερταιν νυμβερ. Ωηατ νυμβερ μακεσ ιτ βαλανχε?

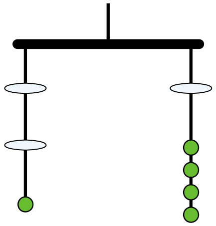


3 Τηισ μοβιλε *never balances* νο ματτερ ωηατ νυμβερ τηε βυχκετ ρεπρεσεντσ. Ωηψ?



4 Δοεσ τηισ μοβιλε βαλανχε *sometimes, always, or never?*

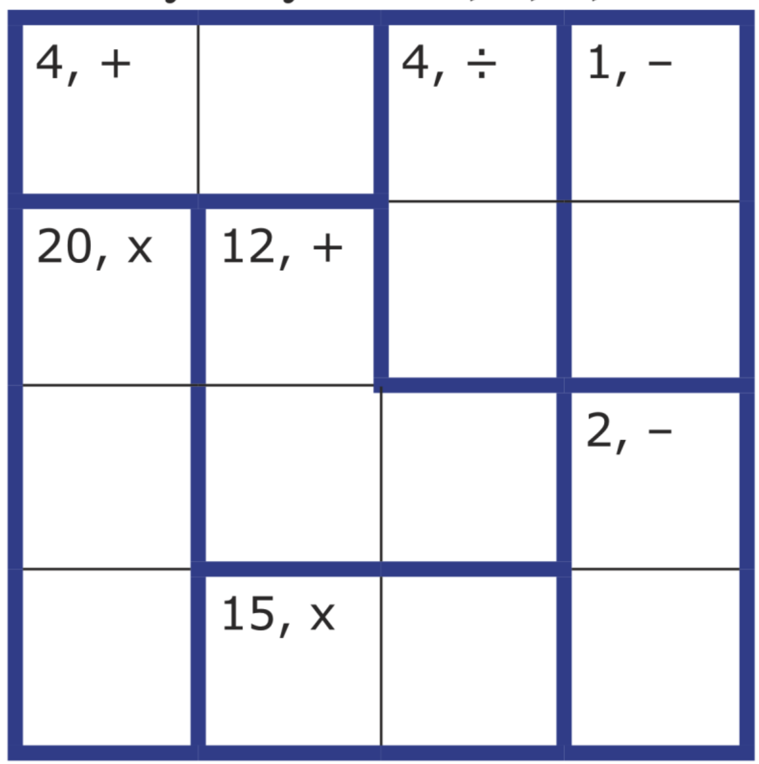
Ιφ σομετιμεσ, *when?*



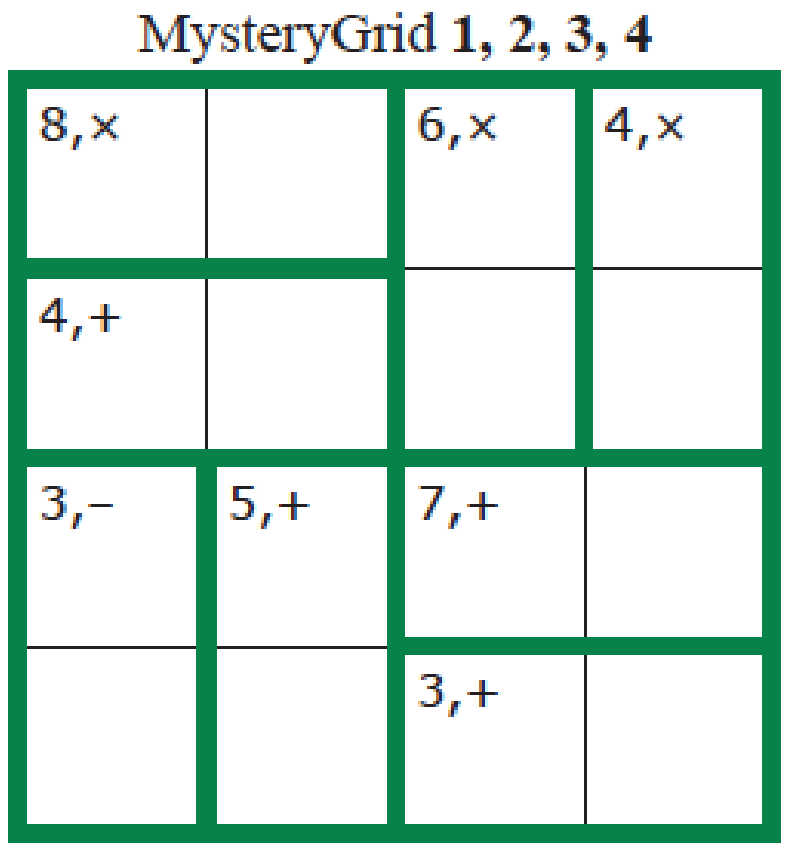
MysteryGrid Puzzles

Find puzzles like these on kenken.com

MysteryGrid **1, 3, 4, 5**

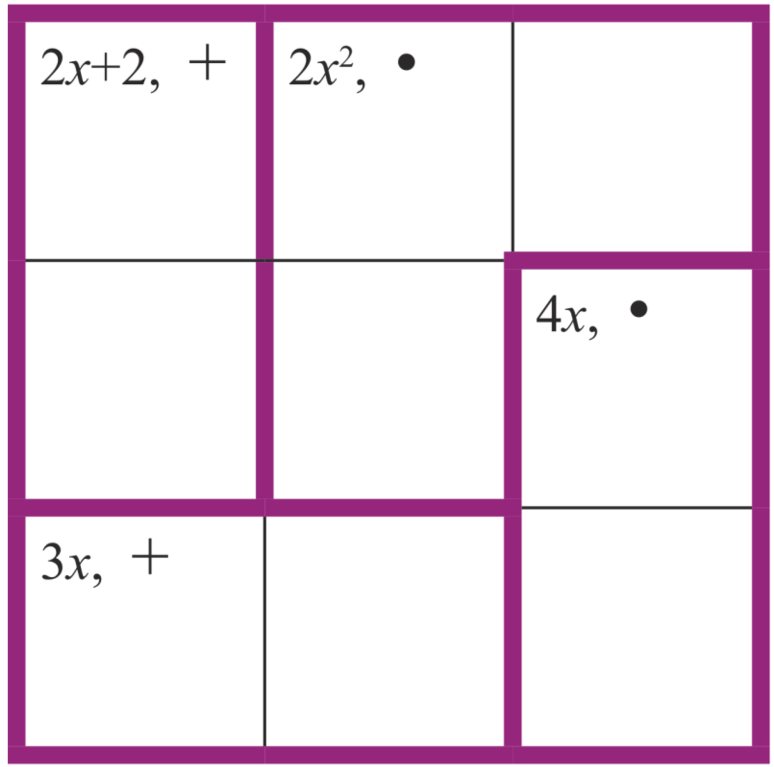


MysteryGrid Puzzles



MysteryGrid Puzzles

MysteryGrid **2, *x*, 2*x***

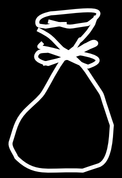


A number trick

- *Think* of a number.
- Add 3.
- Double the result.
- Subtract 4.
- Divide the result by 2.
- Subtract the number you first thought of.
- Your answer is 1!

How did it work?

- *Think* of a number.



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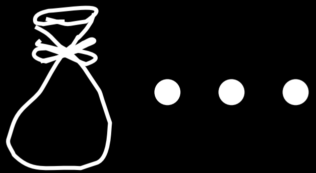

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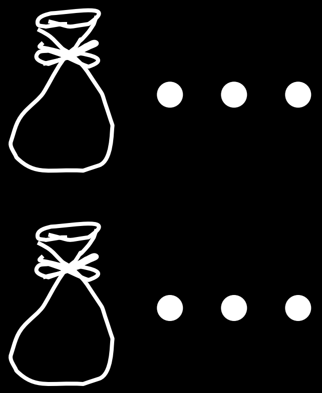

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- Add 3.
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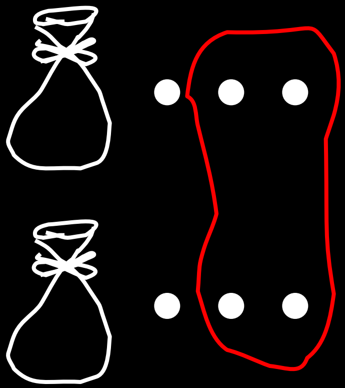
- Subtract 4.



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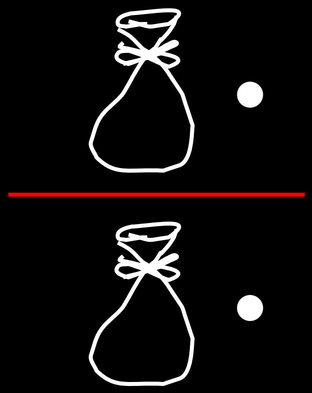

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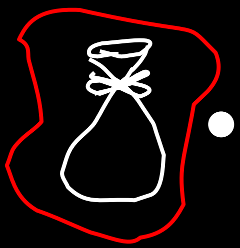

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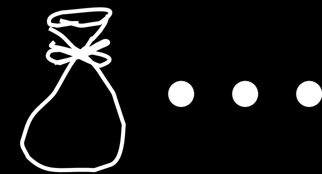
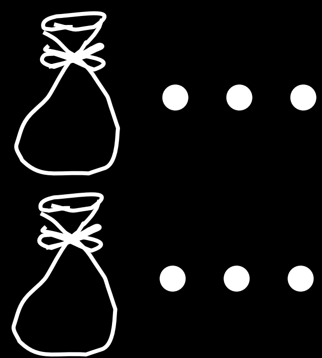
- Your answer is 1!

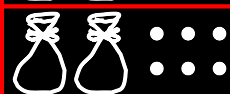
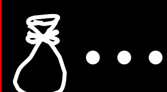
The distributive property before multiplication!

- When you described doubling 
- To make 
- You were implicitly using the distributive property. *Kids* do that, too, before they learn the property in any formal way.

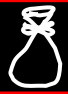
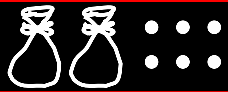
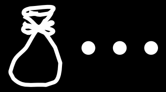
Back to the number trick.

Kids need to do it themselves…

Using notation: *following* steps

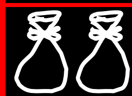
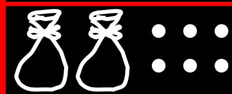
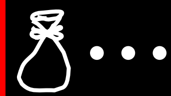
Words	Pictures	Imani	Cory	Amy	Chris
Think of a number.		5			
Double it.		10			
Add 6.		16			
Divide by 2. What did you get?		8	7	3	20

Using notation: *undoing* steps

Words	Pictures	Imani	Cory	Amy	Chris
Think of a number.		5			
Double it.		10			
Add 6.		16	14		
Divide by 2. What did you get?		8	7	3	20

Hard to undo using the words.
Much easier to undo using the notation.

Using notation: *simplifying* steps

Words	Pictures	Imani	Cory	Amy	Chris
Think of a number.		5	4		
Double it.		10			
Add 6.		16			
Divide by 2. What did you get?		8	7	3	20

Abbreviated *speech*: simplifying *pictures*

Words	Pictures	Imani	Cory	Amy	Chris	
Think of a number.		5	4			b
Double it.		10				$2b$
Add 6.		16				$2b+6$
Divide by 2. What did you get?		8	7	3	20	$b+3$

Even practice can be can be mathematical

- Two examples:
  - “Drill and thrill”
  - Discovering a surprising multiplication fact

What about *fact practice*?

Surgeon general’s warning:

- Drill (done wrong) can be hazardous to your health.
- It puts brains to sleep.
- Salience is often low.
- Emotional impact is often tension, which *reduces* performance.
- *Any repetitive exercise*, even physical, can hurt if it’s done wrong.

But we *know* practice *can* help

- Piano or violin, soccer or baseball, handwriting or reading

So what makes *good* practice?

Repetition has been studied (tons!)

Drill doesn't have to kill.

Drill and thrill!

Find this video on thinkmath.edc.org

Exit

What makes something memorable?

- Highly focused and *not **just** memory*
- Intellectually surprising or enlivening
- Brief
- Distributed (10 separated one-minute sessions better than 1 ten-minute session)
- Lively but not pressured
- Reappearing in new contexts
- Letting people *feel* their competence

And about *structure*

- Two brief videos

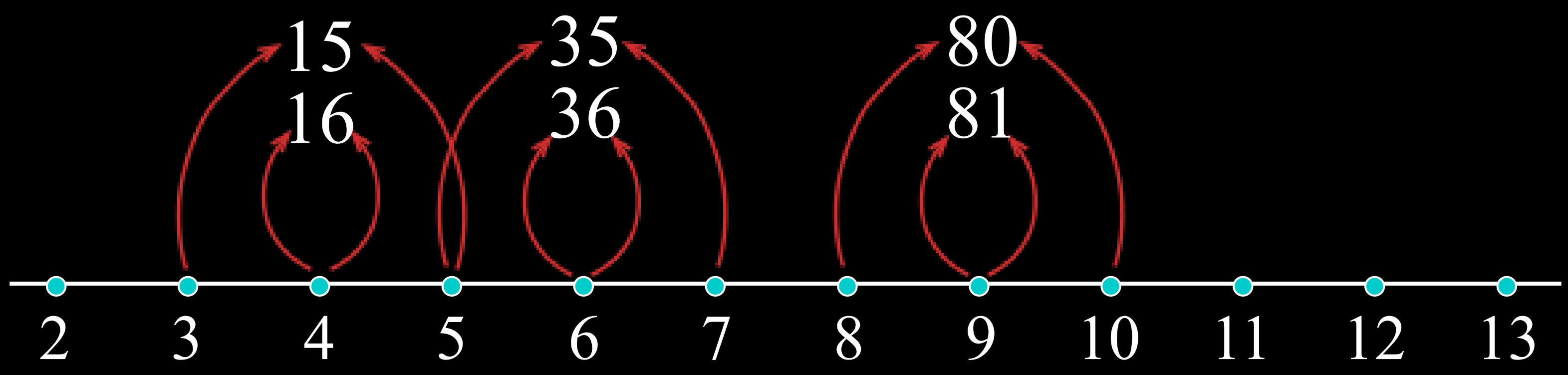
Find this video on
thinkmath.edc.org
Professional development

http://thinkmath.edc.org/index.php/Professional_development_topics

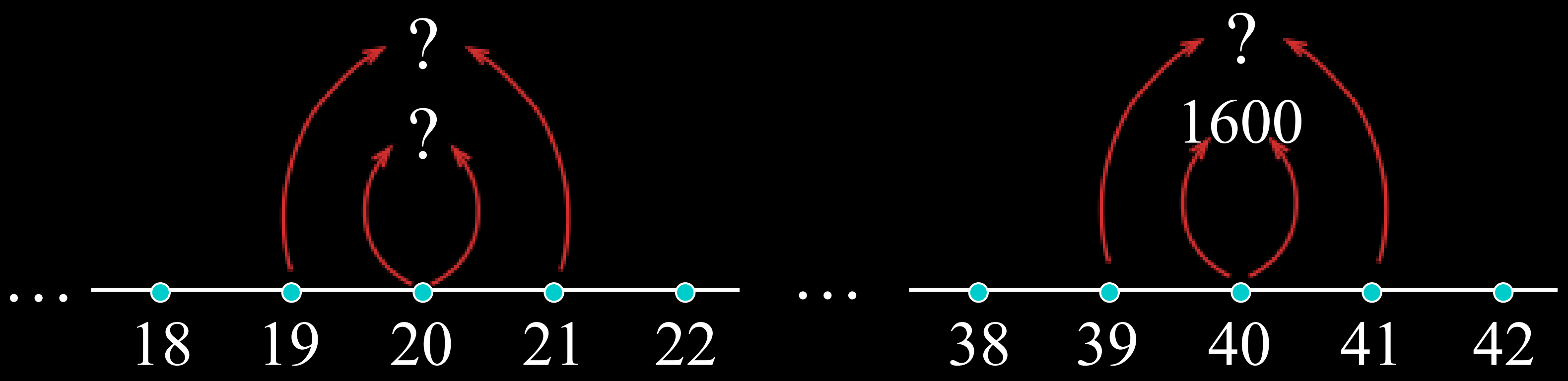
Exit

Teaching without talking

Shhh… Students thinking!



Wow! Will it always work? Big numbers?



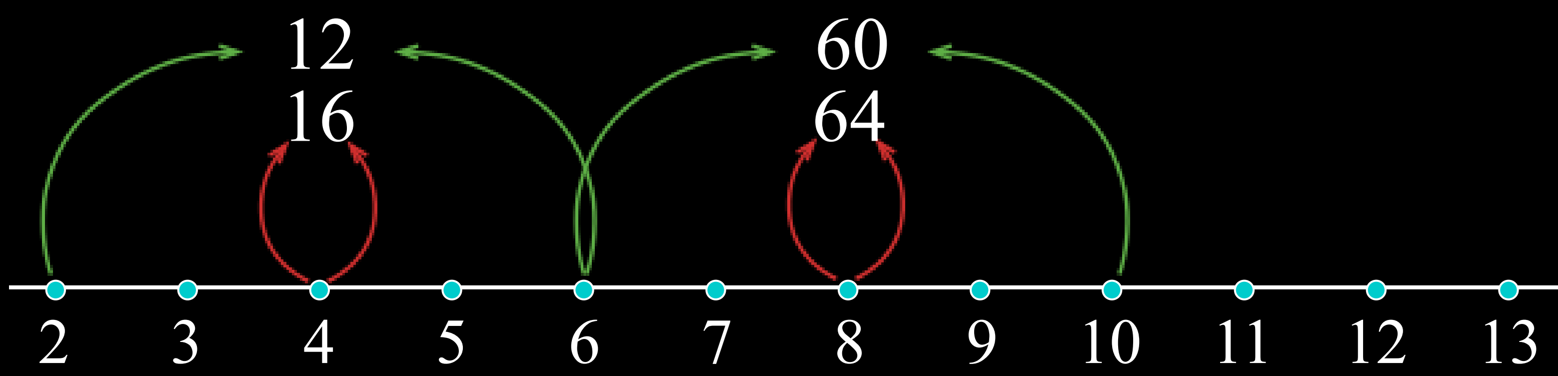
Take it a step further

What about <u>two</u> steps out?

Exit

Teaching without talking

Shhh… Students thinking!



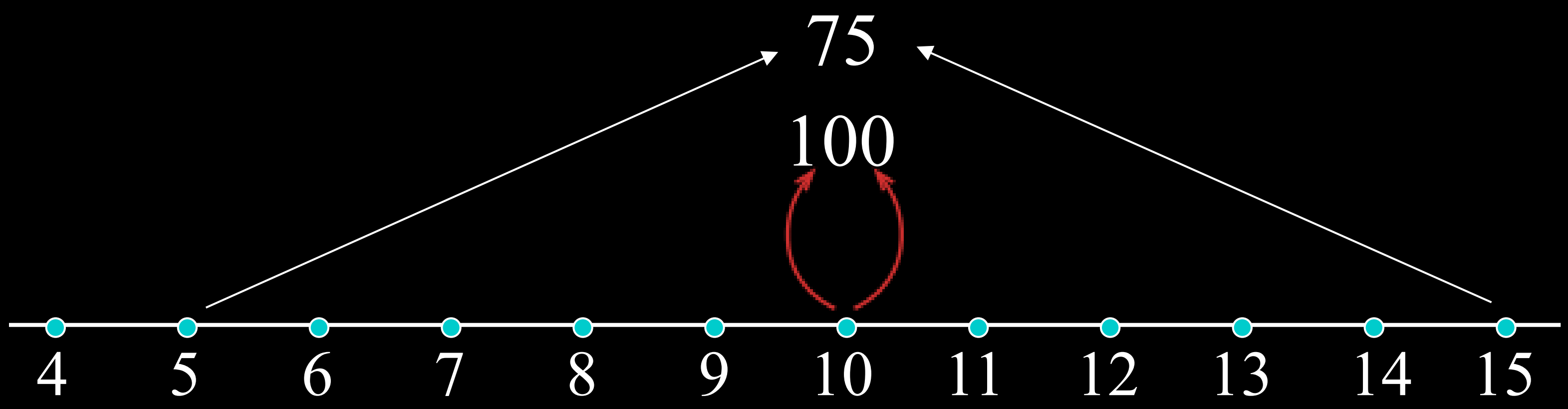
Again?! Always? Find some bigger examples.

Take it even further

What about three steps out?

What about four?

What about five?



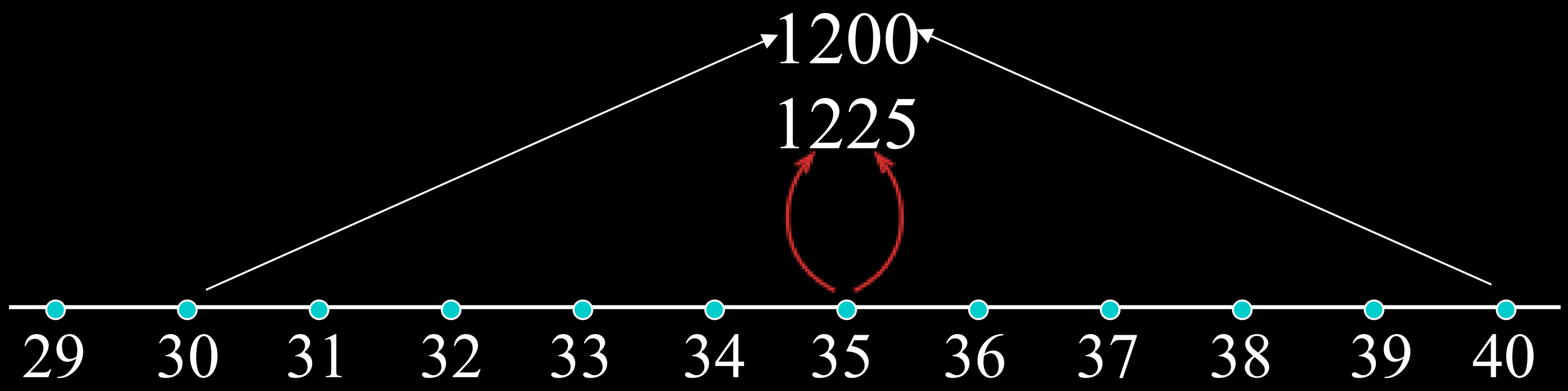
Exit

Take it even further

What about three steps out?

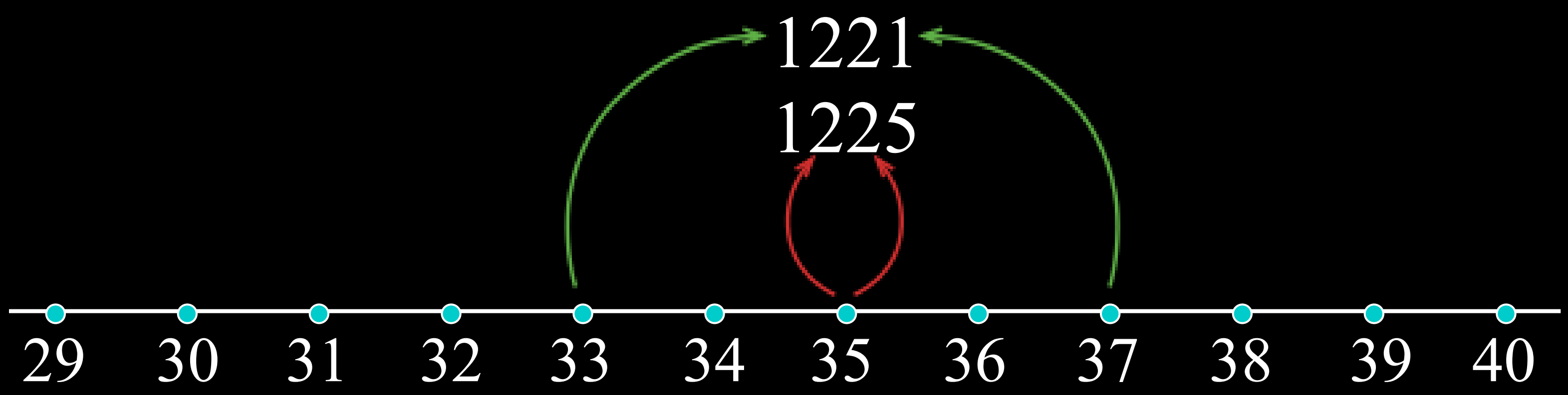
What about four?

What about five?



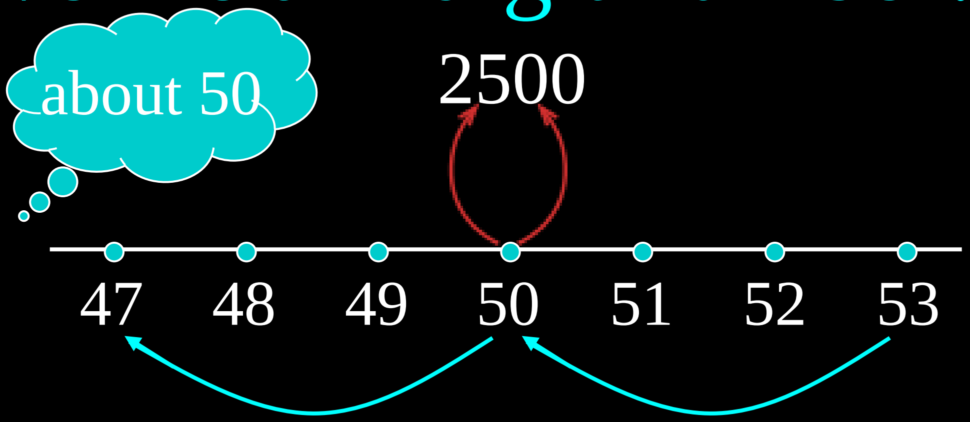
Take it even further

What about two steps out?



“*Mommy!* Give me a 2-digit number!”

- “OK, um, 53”
- “Hmm, well…



- …OK, I’ll pick 47, and I can multiply those numbers faster than you can!”

To do…

$$\begin{array}{r} 53 \\ \times\ \ 47 \\ \hline \end{array}$$

I think…

50 × 50 (well, 5 × 5 and …)… 2500

Minus 3 × 3 – 9

2491

But *nobody cares* if kids can multiply 47×53 mentally!

What *do* we care about, then?

- 50 × 50 (well, 5 × 5 and place value)
- Keeping 2500 in mind while thinking 3 × 3
- Subtracting 2500 – 9
- Finding the pattern
- *Describing* the pattern

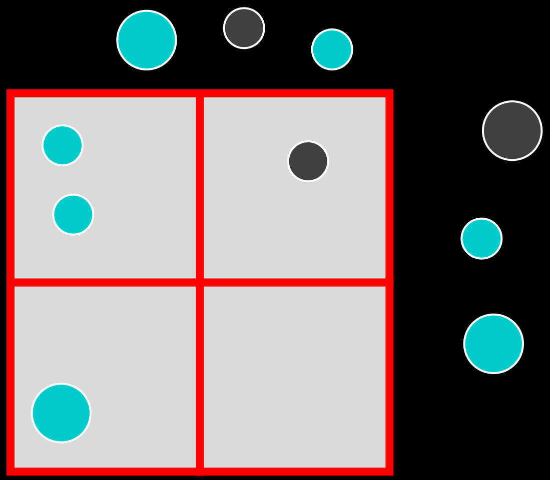
Algebraic/arithmetic thinking

Science

Exit

Algebraic language

Sorting in Kindergarten

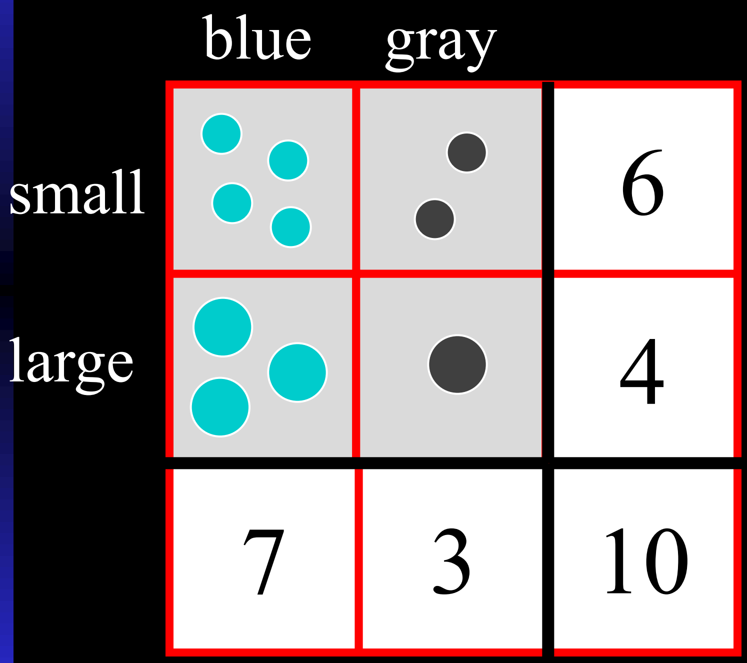


Picture a young child with a small pile of buttons.

Natural to sort.

We help children refine and extend what is already natural.

Back to the very beginnings

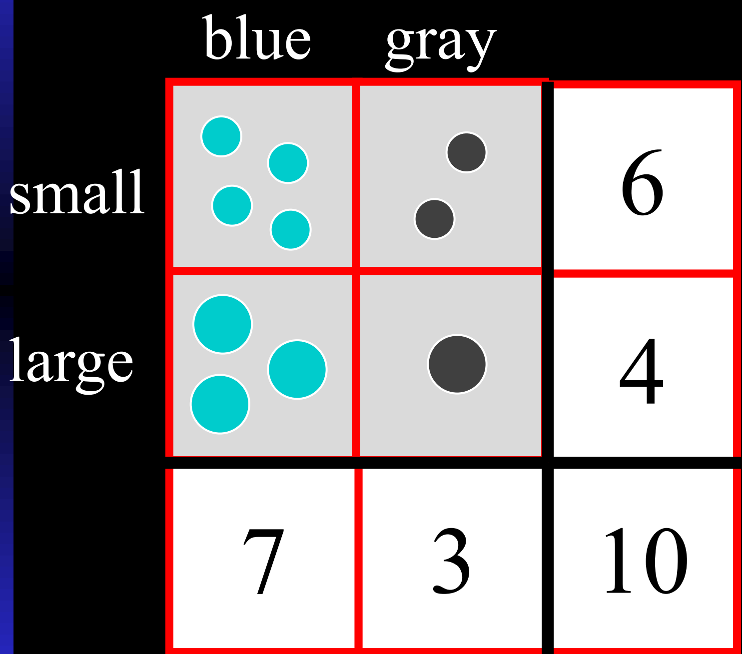


Children can also summarize.

“Data” from the buttons.

Abstraction

If we substitute numbers for the original objects…



4	2	6
3	1	4
7	3	10

A Cross Number Puzzle

Don't always start with the question!

7	6	13
5	3	8
12	9	21

Building the addition algorithm

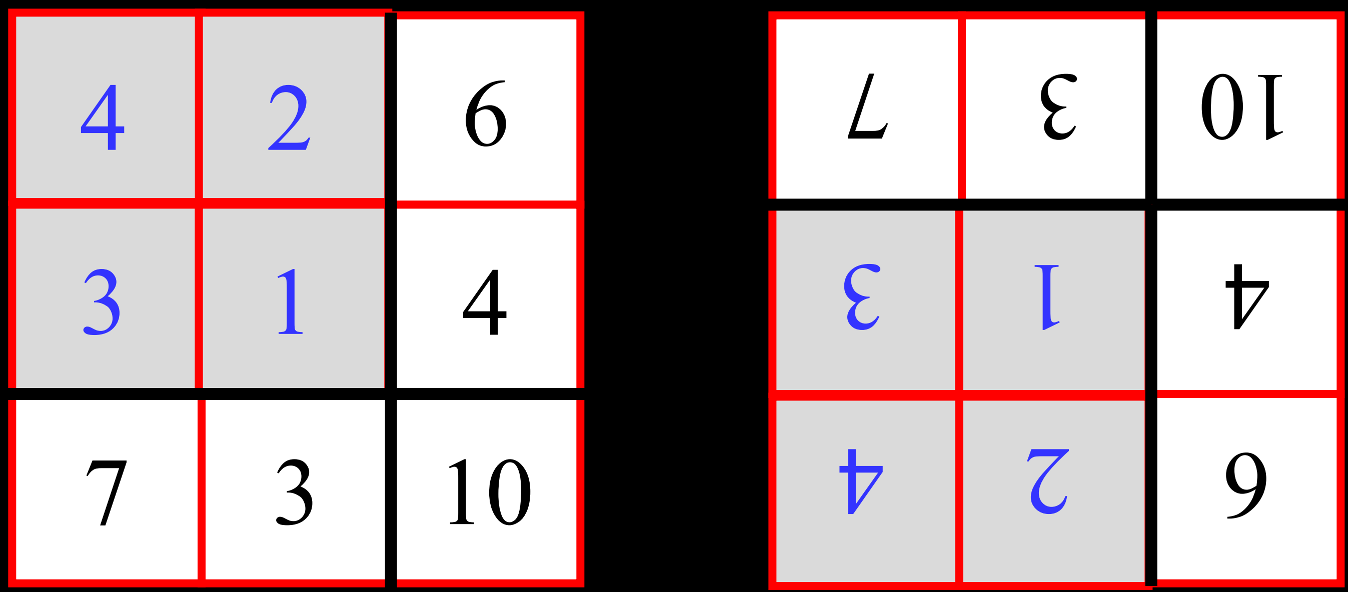
Only multiples of 10 in yellow. Only less than 10 in blue.

20	5	25
30	8	38
50	13	63

$$\begin{array}{r} 25 \\ +\ 38 \\ \hline 63 \end{array}$$

50 + 13 = 63

Relating addition and subtraction



Ultimately, building the addition and subtraction algorithms

The addition algorithm

Only multiples of 10 in yellow. Only less than 10 in blue.

20	5	25
30	8	38
50	13	63

$25 + 38 = 63$

The subtraction algorithm

Only multiples of 10 in yellow. Only less than 10 in blue.

20	5	25
30	8	38
50	13	63

25 + 38 = 63

60	3	63
30	8	38
30	–5 Uh oh!	25

63 – 38 = 25

The subtraction algorithm

Only multiples of 10 in yellow. Only less than 10 in blue.

20	5	25
30	8	38
50	13	63

25 + 38 = 63

50 / 60	13 / 3	63
30	8	38
20	5	25

63 – 38 = 25

The subtraction algorithm

Only multiples of 10 in yellow. Only less than 10 in blue.

20	5	25
30	8	38
50	13	63

$25 + 38 = 63$

53	10	63
30	8	38
23	2	25

$63 - 38 = 25$

The algebra connection: adding

4	2	6
3	1	4
7	3	10

$4 + 2 = 6$

$3 + 1 = 4$

$7 + 3 = 10$

The algebra connection: subtracting

7	3	10
3	1	4
4	2	6

$7 + 3 = 10$

$3 + 1 = 4$

$4 + 2 = 6$

The eighth-grade look

$5x$	$3y$	23
$2x$	$3y$	11
$3x$	0	12

$$5x + 3y = 23$$

$$2x + 3y = 11$$

$$3x + 0 = 12$$

$$x = 4$$

All from buttons!

There's always more, of course…

Thank you!

The Number Line
is special

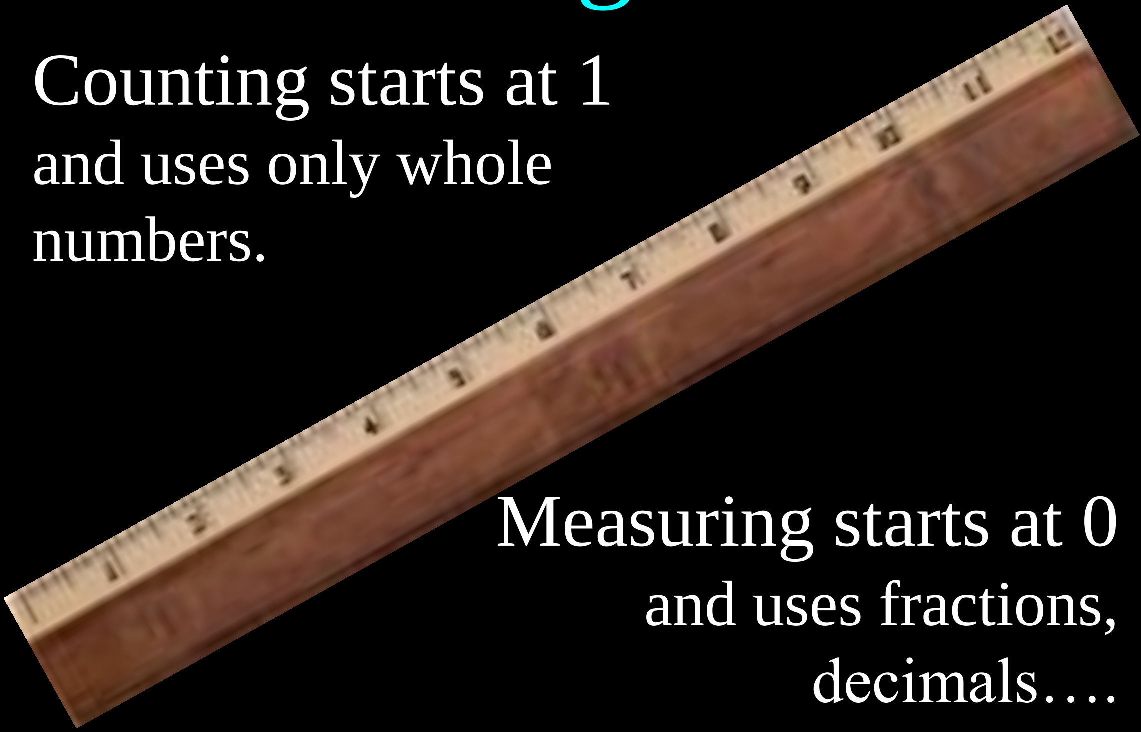
- Unlike Cuisenaire rods, base 10 blocks…,   the number line is *not* just a school tool: mathematicians use it, too.
- *All* the numbers that students meet K–10 live on this line: counting numbers, zero, fractions, decimals, negatives.

Counting and measuring are different!

- We use numbers for counting and measuring
- Counting starts at 1, uses only "counting numbers"
  (no fractions, decimals, negatives)
- Measuring starts at 0. It uses *all* numbers.

Counting and measuring are different!

Counting starts at 1
and uses only whole
numbers.

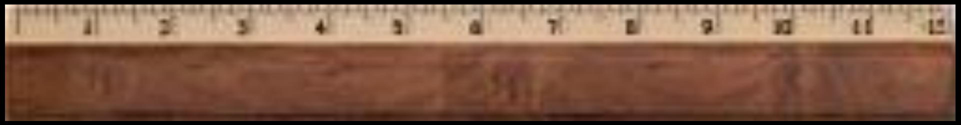


Measuring starts at 0
and uses fractions,
decimals….

- Counters, base-10 blocks, fingers support *counting*, but not measuring.
- The number line supports measuring.

Number lines represent measuring

- The number line is like a ruler

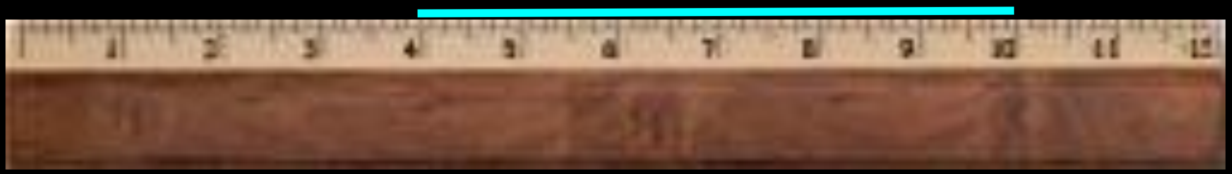


- Numbers are *addresses* on a ruler
- They also name their distance from zero

  Distance is an important meaning of numbers
- *Going* a distance “is” addition
- *Finding* a distance *between* numbers “is” subtraction!
- Going a distance to the right or left “is” addition or subtraction.

How long is this line segment?

- We're not using the ruler "correctly," but we can still figure out how long the line really is.



- The *distance* between 4 and 10 is the length.
- This idea will make sense of computations with whole numbers, fractions, decimals and even negative numbers.

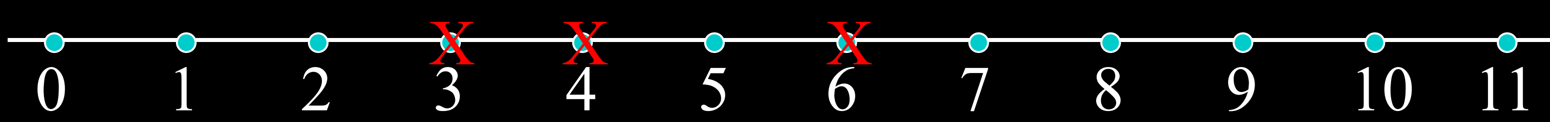
One image of + and –

- A child says where bunny starts
- Another says how far.
- Another says which way.

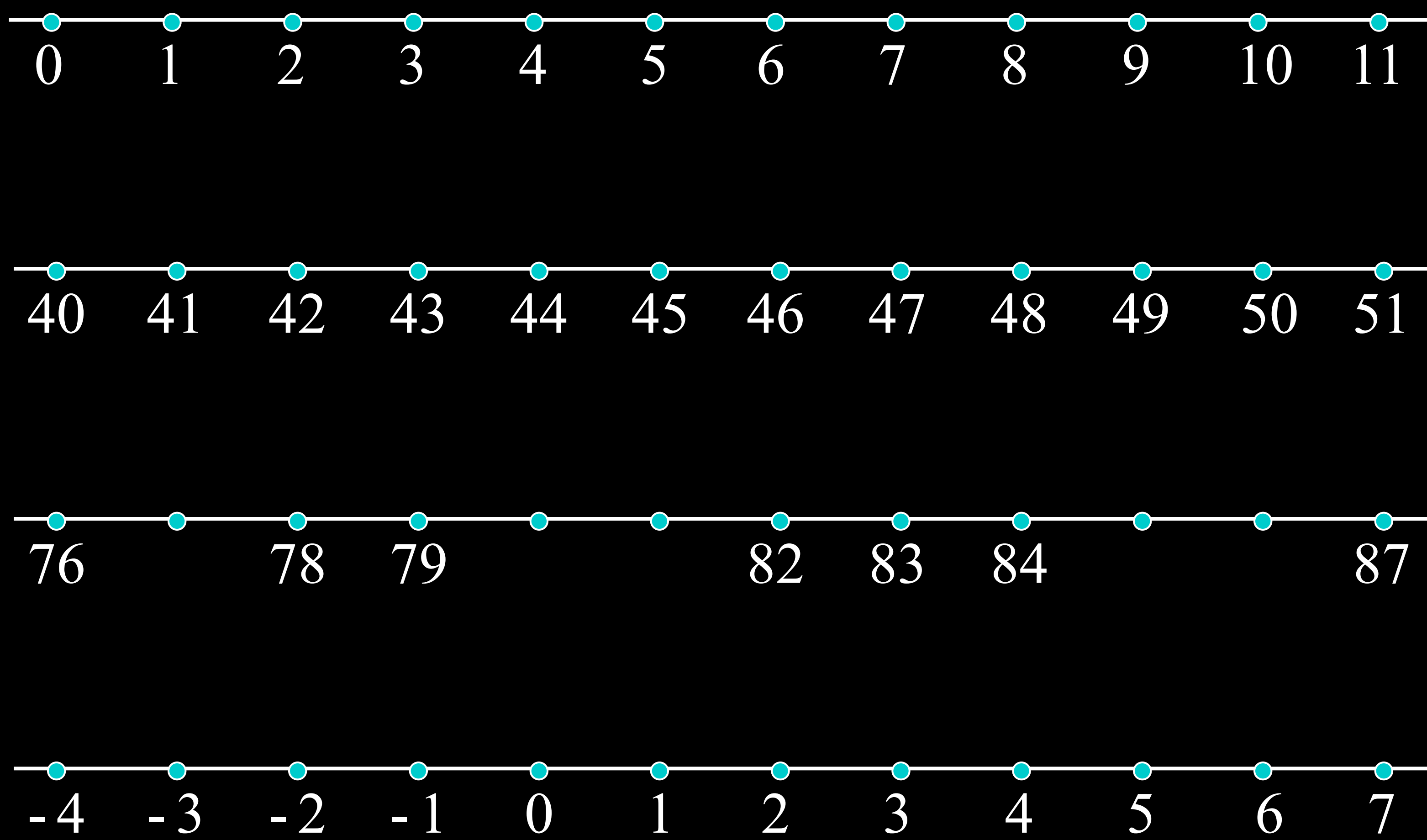
  + 3

  – 2

- Where does bunny land?



Number line segments



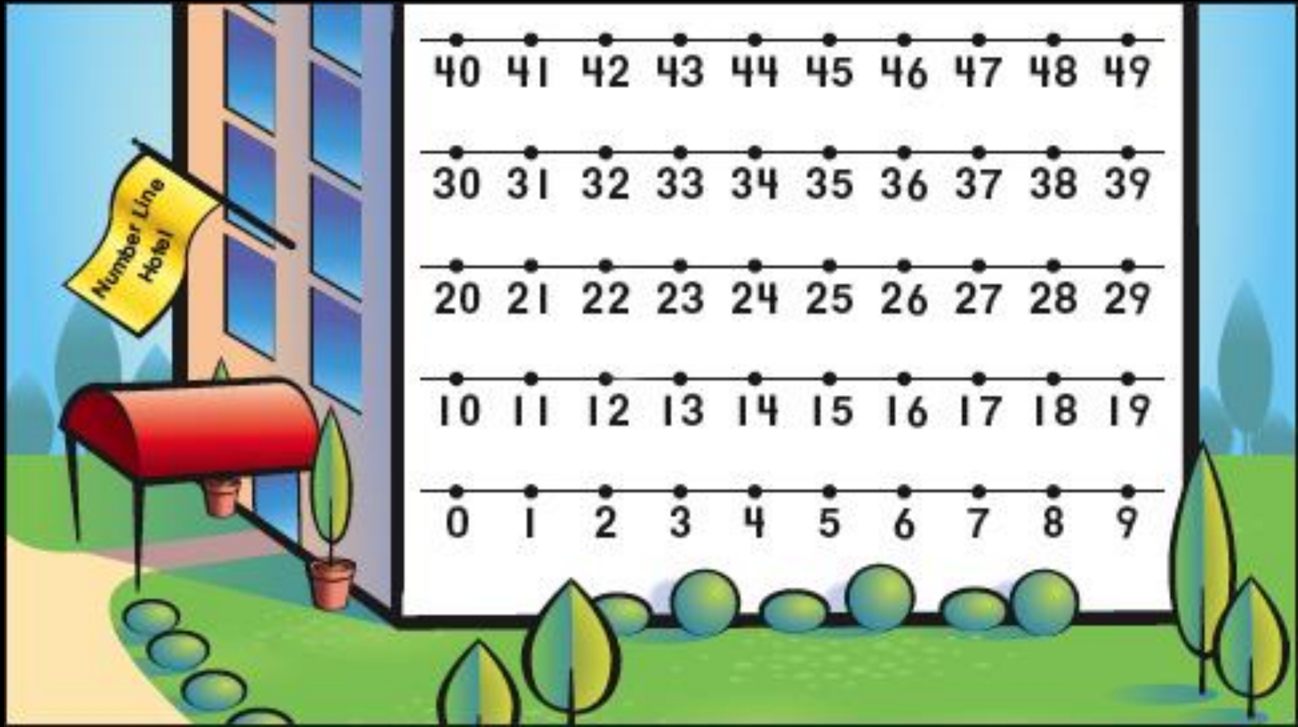
Conventional hundreds chart

- Which is lower, 47 or 37?
- What is 10 higher than 23?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70

“Right side up”

- Which is lower, 47 or 37?
- What is 10 higher than 23?



Conventional hundreds chart

- How are the numbers in one column alike?
- How are the numbers in one row alike?

1	2	3	4	5	6	7	8	9	10
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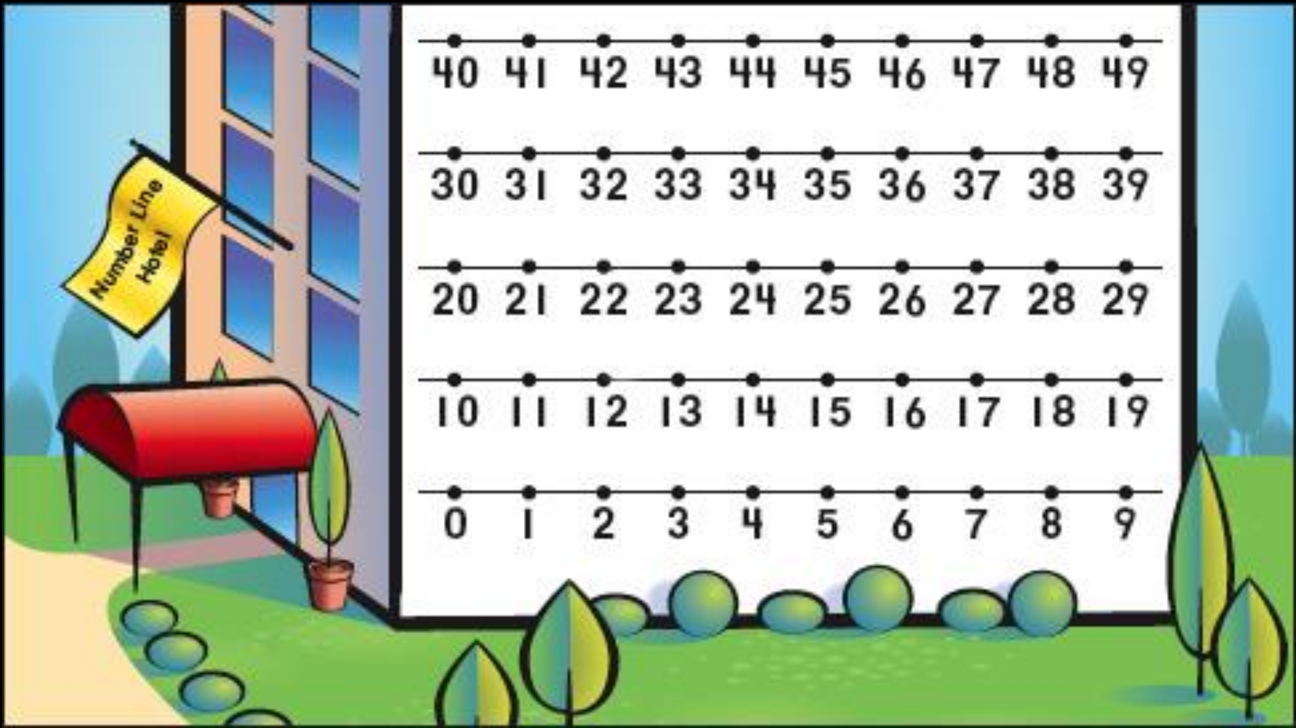
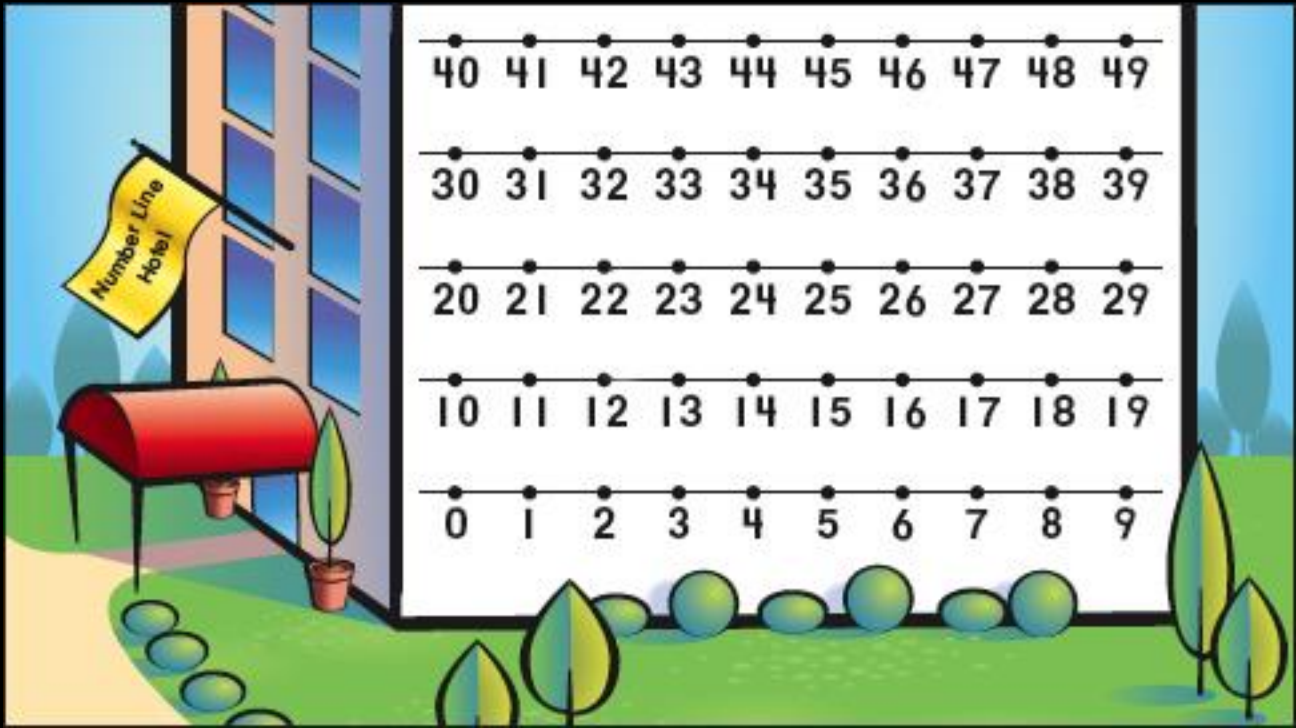


“Right side up”

- Where are the 40s?



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Let's *use* the structure

- How many fingers *don't* you see?
- Pairs to 10
- Dimes --- same structure

What makes something memorable?

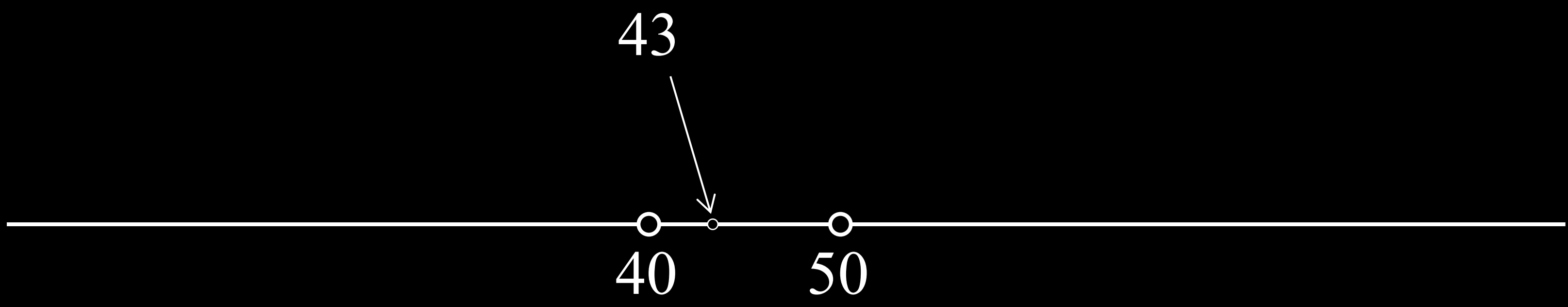
Emotional impact

Intellectual surprise: Aha! moment

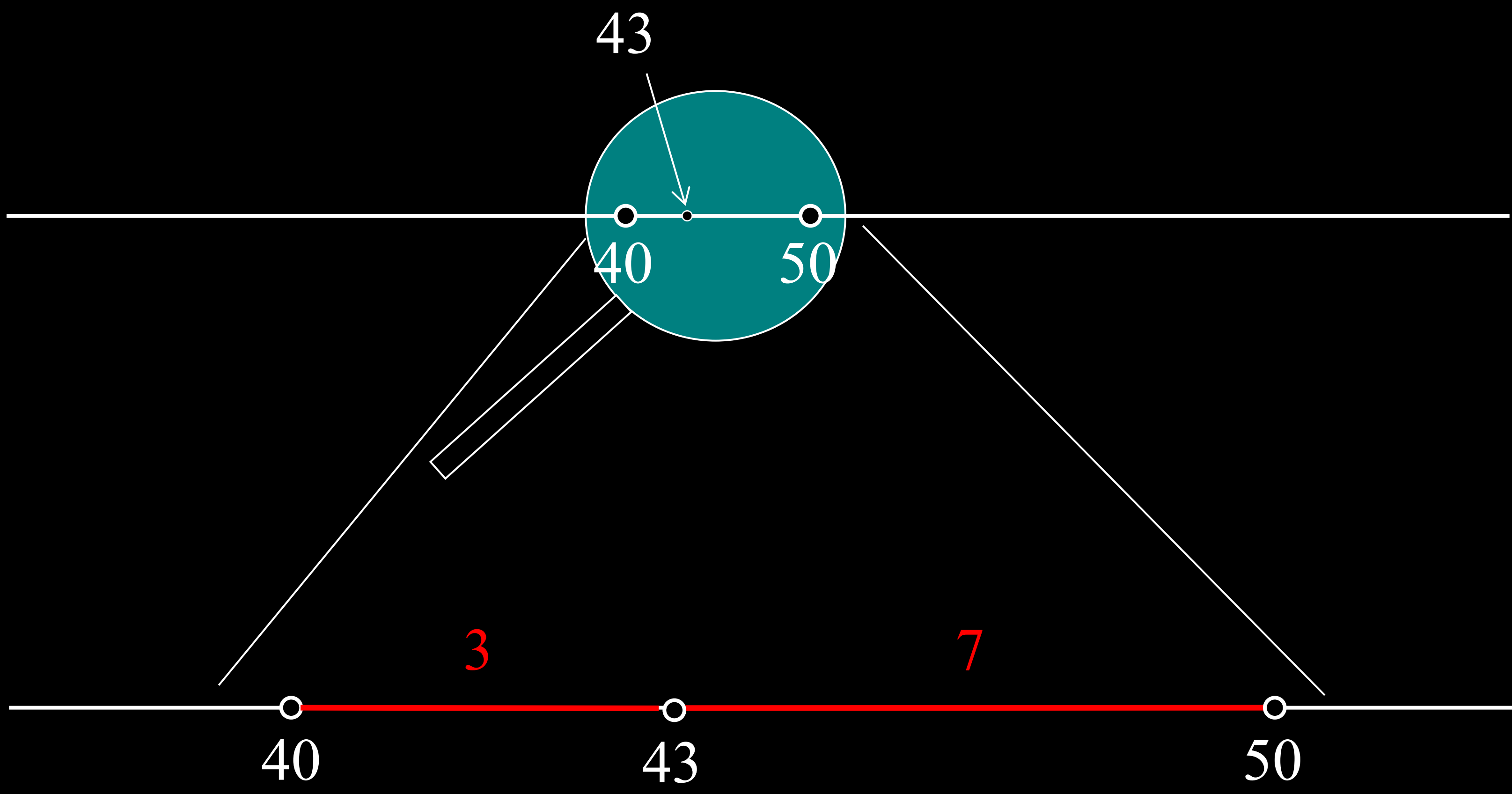
Making sense

Videos

What “tens” is it between?

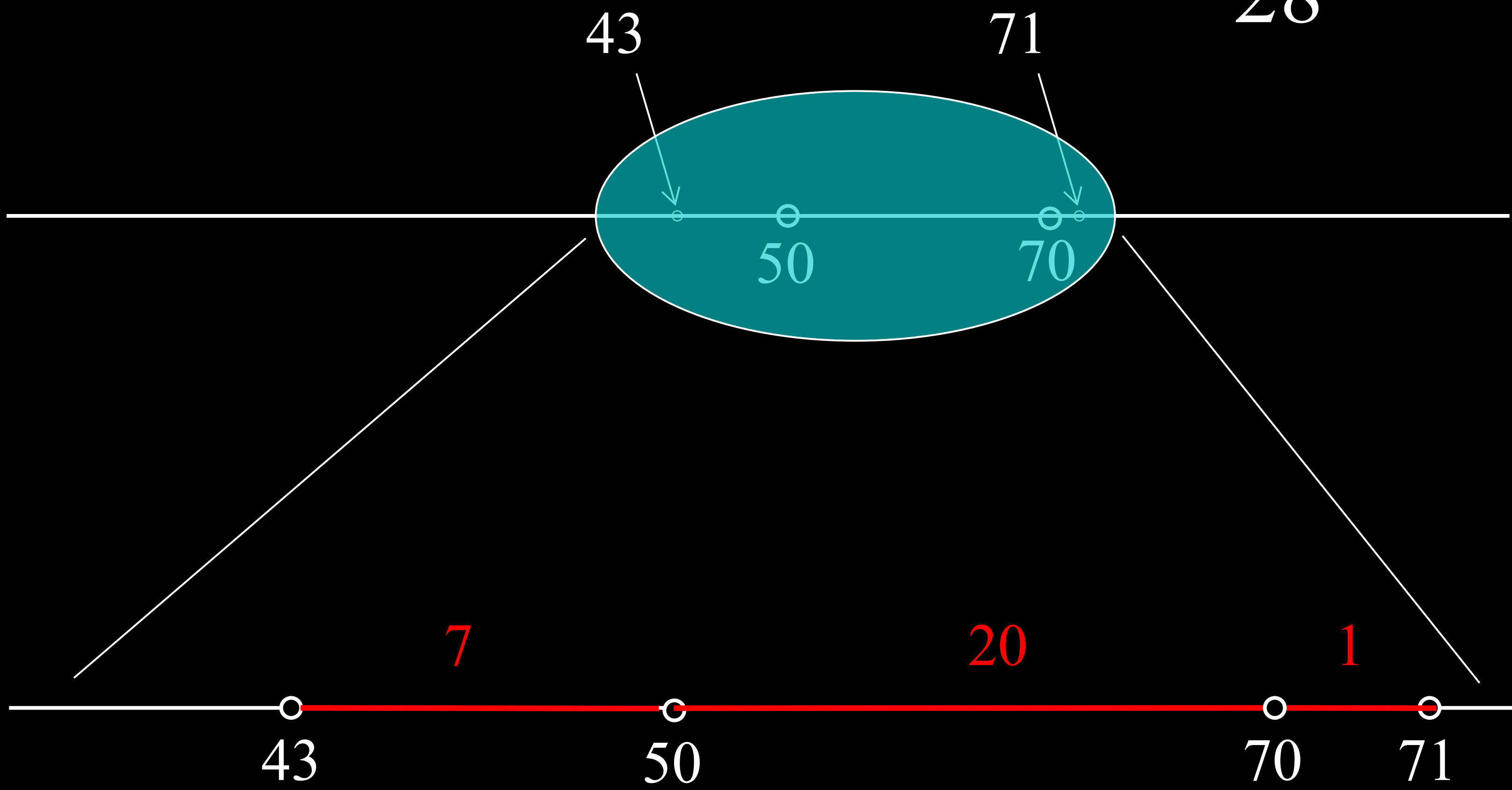


How *far* from the nearest tens?



How far is 43 from 71?

$$\begin{array}{r} 71 \\ -\,43 \\ \hline 28 \end{array}$$

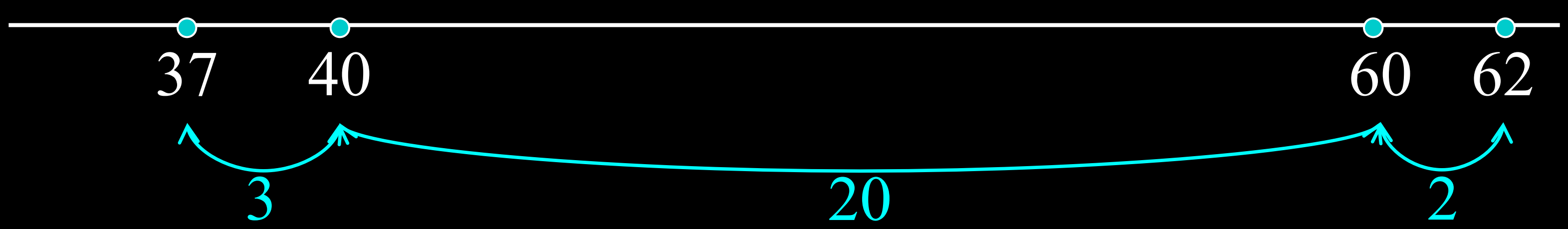


Another example of subtraction

I have saved \$37, but I need \$62 to buy the …

- How can we do 62 – 37?
- It means "how far from 37 to 62?"

- Find the nearest multiples of 10 between them



- So 62 – 37 = 25

Positive and negative

How do we make sense of $(^{-}7) - 5$?

- Is it 2? –2? 12? –12?
- Why should it have *any* meaning at all?!

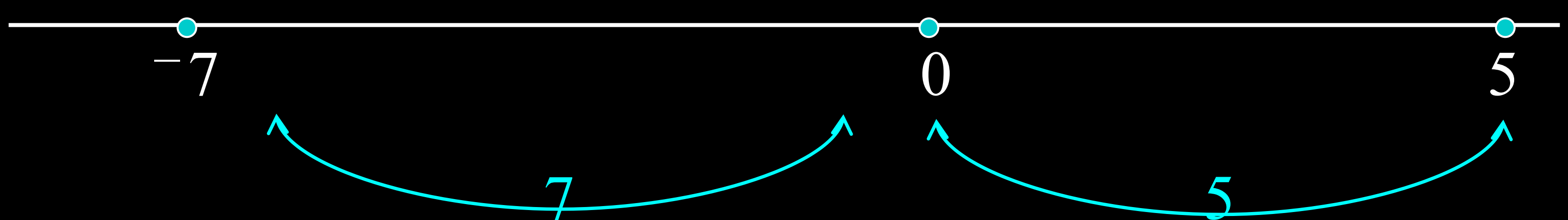
- But if we want to *give* it meaning, we use structure!

Positive and negative

How do we make sense of $(^{-}7) - 5$?

- Think the way we thought about $62 - 37$…
- It means “how far *from* 5 *to* my goal of $^{-}7$?

- We *see* that the distance is (some kind of) 12…



- *…in the negative direction.* So $(^{-}7) - 5 = {}^{-}12$

Positive and negative

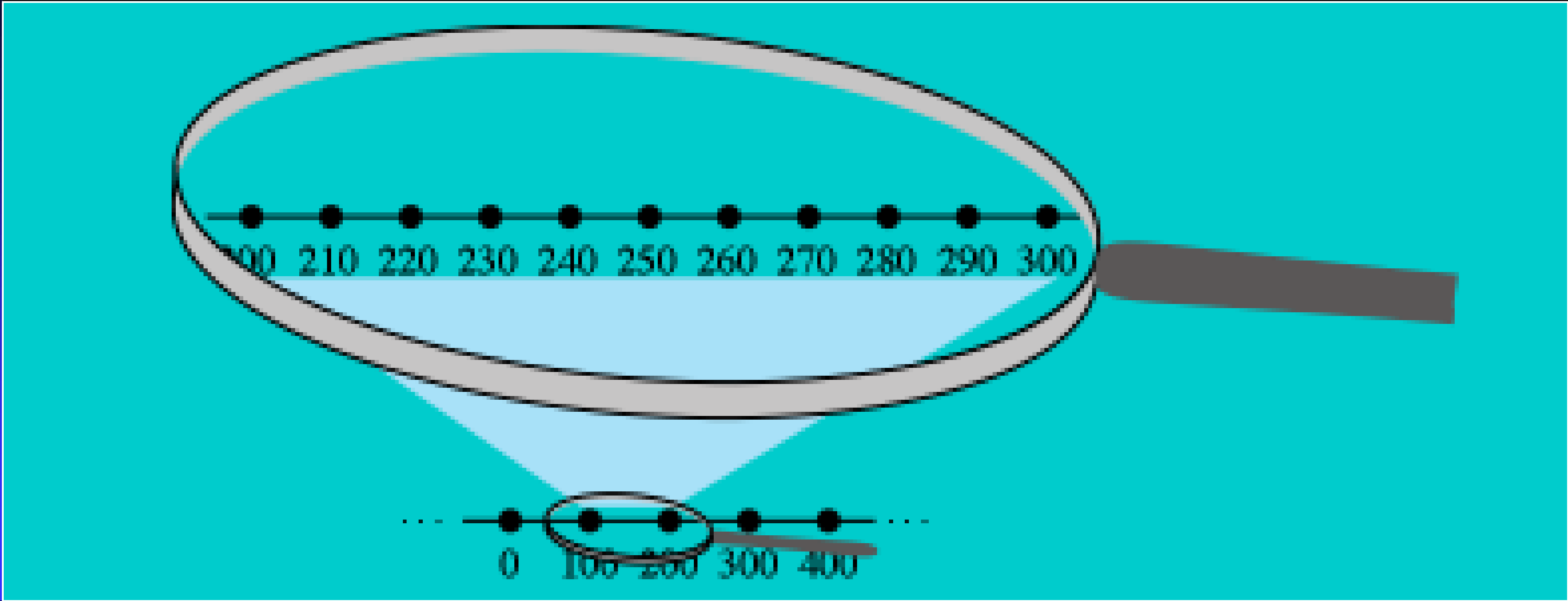
How do we make sense of $(^-5) - (^-7)$?

- The same image works. No "new rule" to learn
- Again, think about $62 - 37$…
- Our problem means "how far *from* $^-7$ *to* $^-5$?"

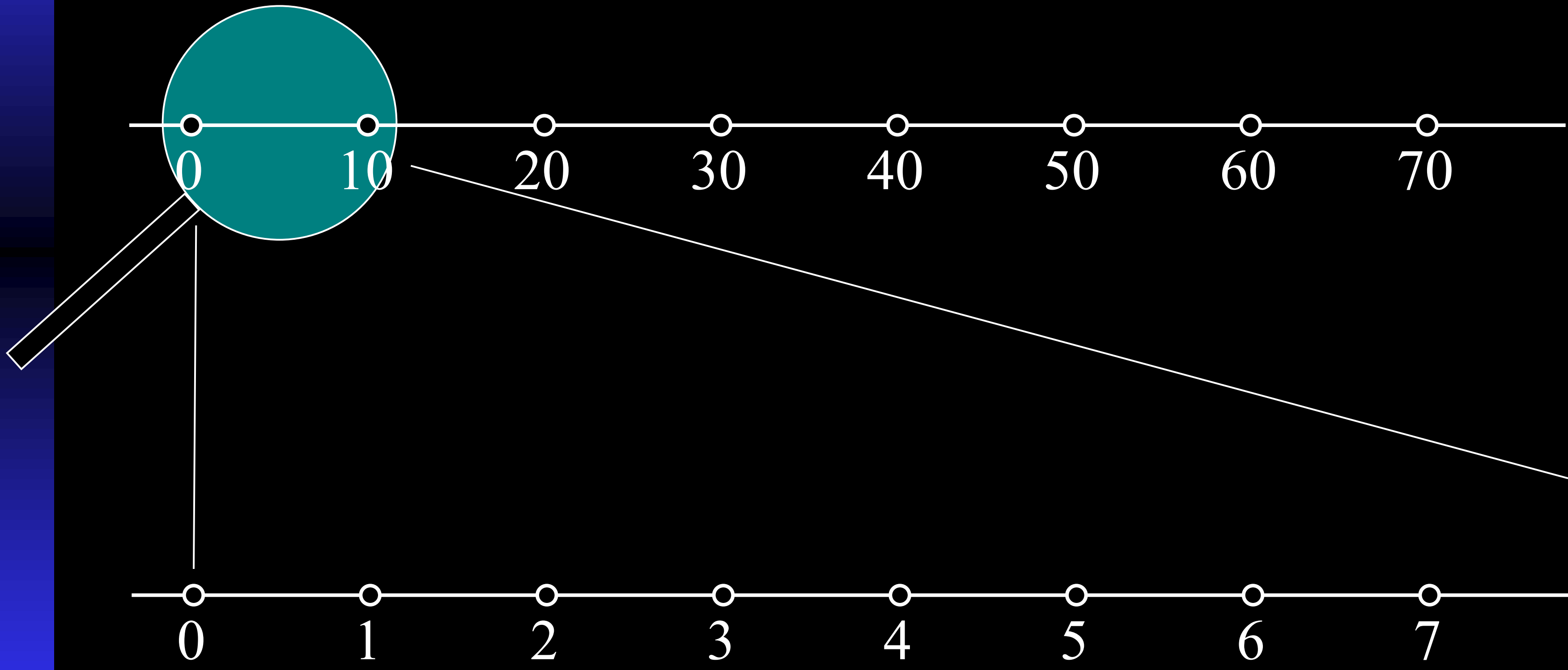
- We *see* that the distance is 2.
- …and *from* $^-7$ *to* $^-5$ *is positive*. So $(^-5) - (^-7) = 2$

Numbers live on a number line…

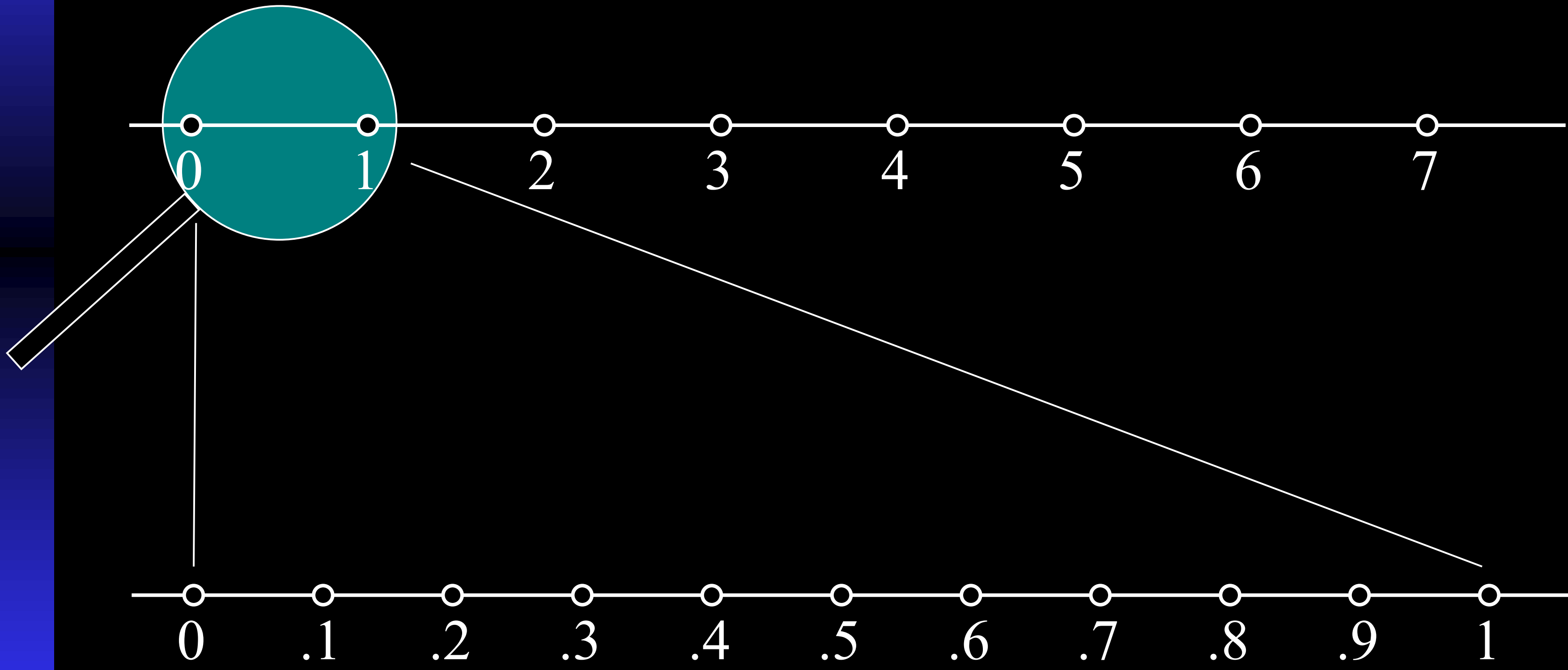
- Numbers serve as location and as distance
- Addition is adding distances
- Subtraction measures the distance between



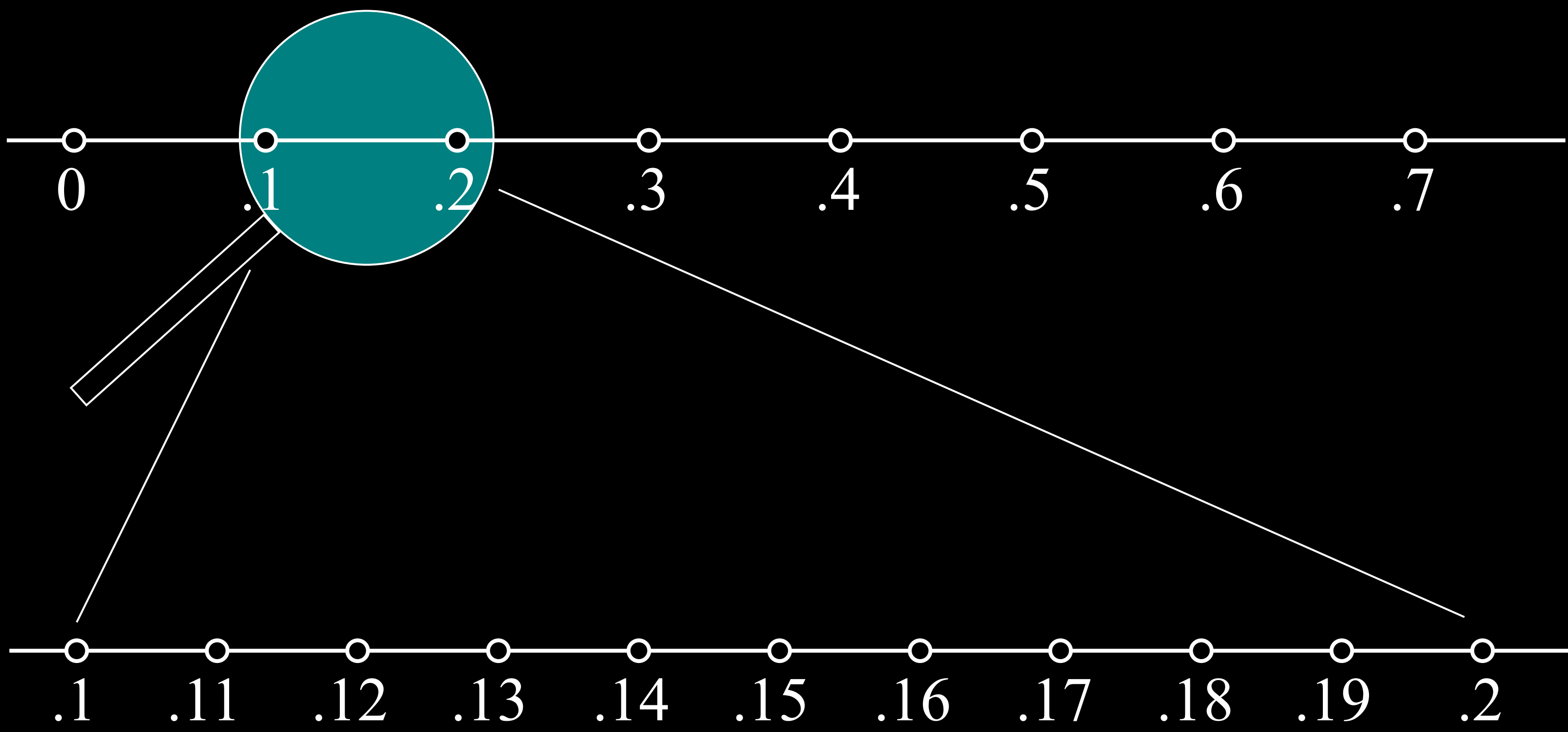
Zooming In



Zooming In More

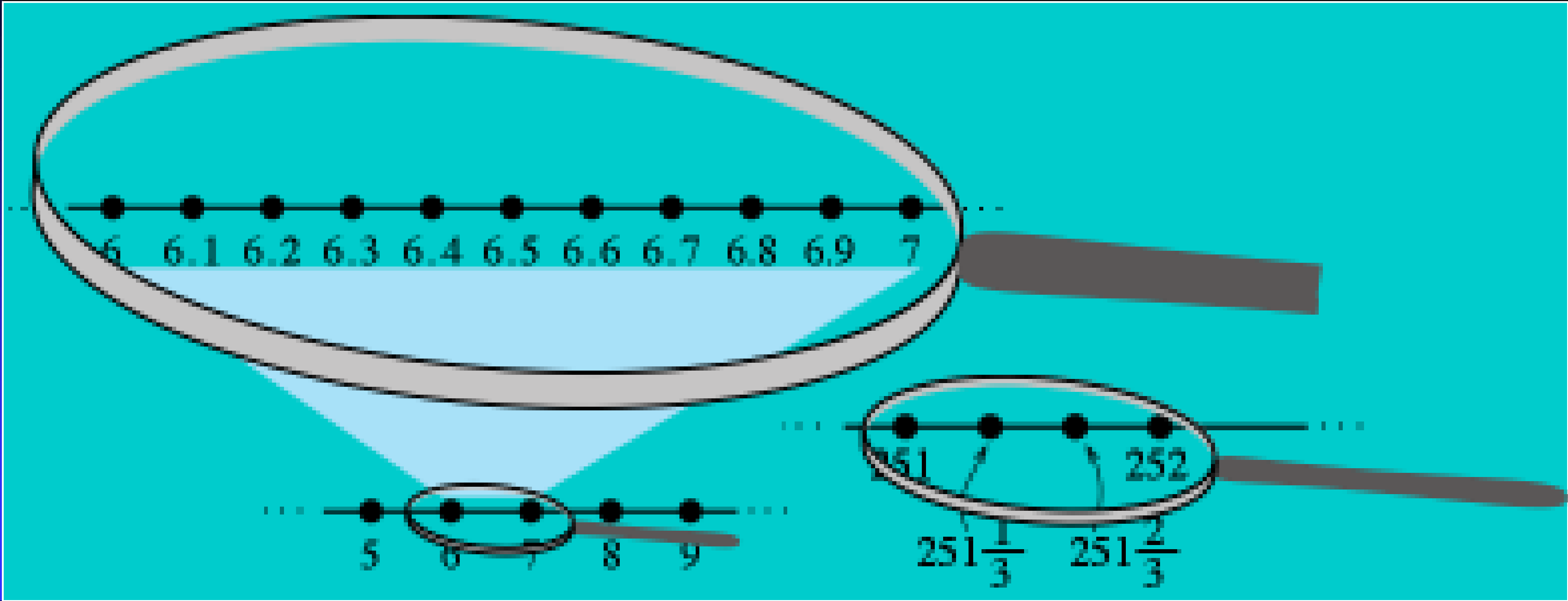


Zooming In Still More



All numbers live on a number line!

- Numbers serve as location and as distance
- Addition is adding distances
- Subtraction measures the distance between
- Fractions live there, too.



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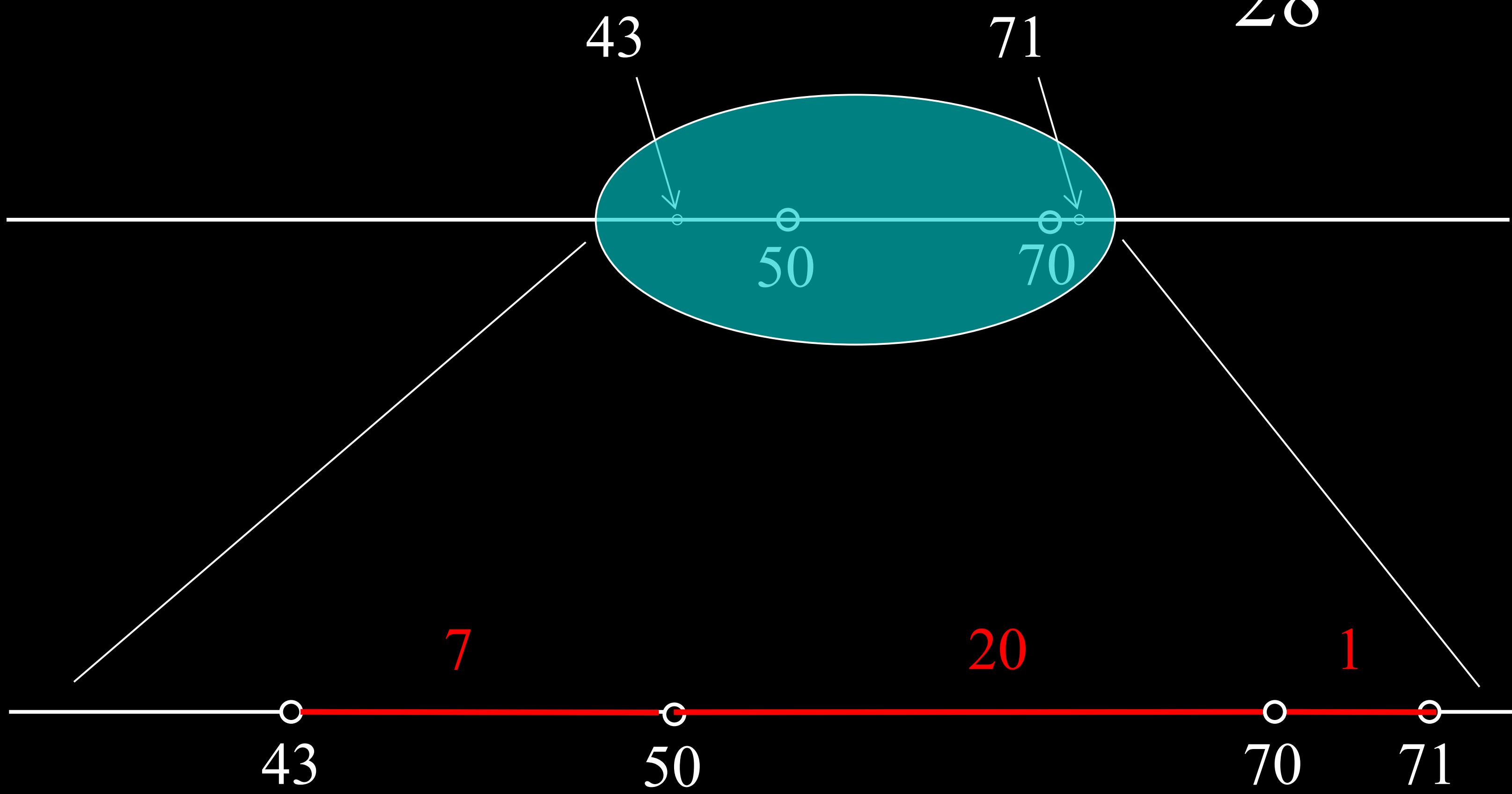
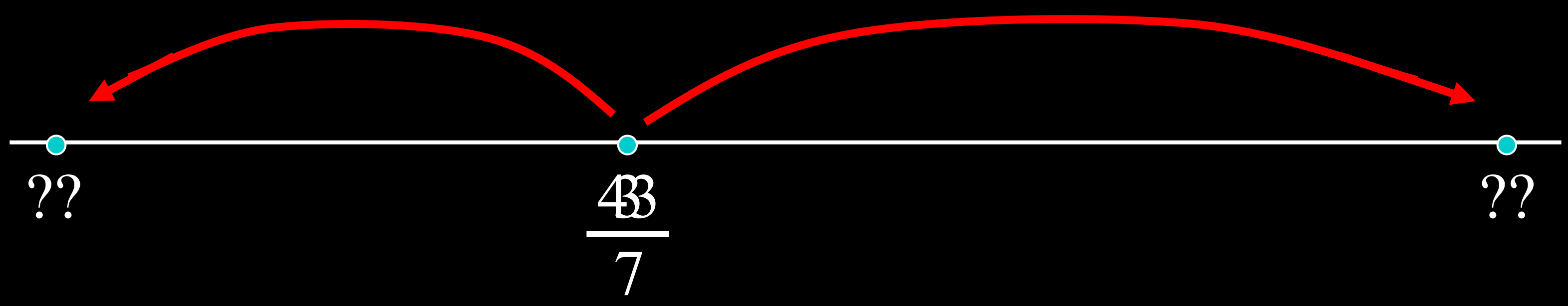


Connect everything to the number line!

Fractions: a familiar question

using what you know about whole numbers

- What are the nearest “tens”?
  How far from each?



- What are the nearest whole numbers?
  How far from each?

Subtracting whole numbers using complements

- Michelle's strategy for 24 – 7:

    (breaking it up)

Algebraic ideas

- Well, 24 – *4* is easy!
- Now, 20 minus *another* 3…
- Well, I know *10* – 3 is 7,

    and 20 is 10 + 10,

  so, *20* – 3 is 17.

Completing 10

- So, 24 – 7 = 17

Subtracting fractions using complements

- Michelle's strategy for $7\frac{2}{5} - 3\frac{4}{5}$

  (breaking it up) Algebraic ideas
- Well, $7\frac{2}{5} - 3\frac{2}{5}$ is *easy*! That's 4.
- Now, 4 minus *another* $\frac{2}{5}$ …
- Well, I know $1 - \frac{2}{5}$ is $\frac{3}{5}$, Completing 1

  so, $4 - \frac{2}{5}$ is $3\frac{3}{5}$
- So, $7\frac{2}{5} - 3\frac{4}{5} = 3\frac{3}{5}$